Ex 1. 5 *Let $p(x, y)$ be given as in the table.*

Find

- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X,Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X;Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*

$X \backslash Y$	0	1	
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
1	0	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{2}{3}$	1

$\mathbb{P}(X=0, Y=0)$

$\mathbb{P}(X=0)$

$\mathbb{P}(Y=1)$

$\mathbb{P}(Y=0) = \mathbb{P}(Y=0, X=0) + \mathbb{P}(Y=0, X=1)$

a)
$$H(X) = -\sum_{x \in \mathcal{X} = \{0,1\}} p_X(x) \cdot \log(p_X(x))$$
$$= -\Big(p_X(0) \cdot \log(p_X(0)) + p_X(1) \cdot \log(p_X(1))\Big)$$
$$= -\left(\frac{2}{3} \cdot \log_2\left(\frac{2}{3}\right) + \frac{1}{3} \cdot \log_2\left(\frac{1}{3}\right)\right)$$
$$= -\left(\frac{2}{3} \cdot (\log_2(2) - \log_2(3)) + \frac{1}{3} \cdot (\log_2(1) - \log_2(3))\right)$$
$$= -\left(\frac{2}{3}(1 - \log_2(3)) + \frac{1}{3}(0 - \log_2(3))\right)$$
$$= -\left(\frac{2}{3} - \log_2(3)\right)$$
$$= \log_2(3) - \frac{2}{3} \approx 0{,}92 > 0$$

$$\boxed{\log\left(\frac{a}{b}\right) = \log(a) - \log(b)}$$

$\log_2(2) = 1$, da $2^{\boxed{1}} = 2$

$\log_2(1) = 0$, da $2^0 = 1$

Mädchen, Jungen, 500 Kinder, Brille oder kein

	M	J	
B	100	50	150
$\bar{B}$	100	250	350
	200	300	500

$$\mathbb{P}(B \mid J) = \frac{50}{300} = \frac{1}{6}$$

$$\mathbb{P}(B \mid M) = \frac{100}{200} = \frac{1}{2}$$

$$\mathbb{P}(M \mid B) = \frac{100}{150}$$

$$\mathbb{P}(J \mid \bar{B}) = \frac{250}{350}$$

miro

$$H(Y) = \log_2(3) - \frac{2}{3}$$

$$H(X|Y) =$$

$$= -\sum_{x,y \in \mathcal{X}\times\mathcal{Y}} p_{X,Y}(x,y)\log(p_{X|Y}(x|y))$$

$$= -p_{X,Y}(0,0)\log(p_{X|Y}(x=0|y=0))$$

$$-p_{X,Y}(1,1)\log(p_{X|Y}(x=1|y=1))$$

$$-p_{X,Y}(1,0)\log(p_{X|Y}(x=1|y=0))$$

$$-p_{X,Y}(0,1)\log(p_{X|Y}(x=0|y=1)) =$$

Ex 1. 5 *Let $p(x, y)$ be given as in the table.*

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find

- *a)* $H(X)$, $H(Y)$
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- *c)* $H(X,Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X;Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*

$$= -\left(\frac{1}{3}\underbrace{\log(1)}_{=0} + \frac{1}{3}\underbrace{\log\left(\frac{1}{2}\right)}_{=-1} + 0 + \frac{1}{3}\underbrace{\log\left(\frac{1}{2}\right)}_{=-1}\right) =$$

$$2^{\boxed{-1}} = \frac{1}{2}$$

$$= \frac{2}{3}$$

miro

Ex 1. 5 *Let $p(x,y)$ be given as in the table.*

$X \backslash Y$	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

Find

- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X,Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X;Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*

c) $$H(X,Y) = -\sum_{(x,y)\in \mathcal{X}\times\mathcal{Y}} p_{X,Y}(x,y)\cdot \log\left(p_{X,Y}(x,y)\right)$$

$$= -\left(\overset{\frac{1}{3}}{p_{X,Y}(0,0)} \cdot \log_2\left(\frac{1}{3}\right) + \overset{\frac{1}{3}}{p_{X,Y}(1,1)}\cdot \log_2\left(\frac{1}{3}\right) + \overset{\frac{1}{3}}{p_{X,Y}(0,1)}\,\log_2\left(\frac{1}{3}\right)\right)$$

$$= -3\cdot\frac{1}{3}\cdot\log_2\left(\frac{1}{3}\right) = -\log_2\left(\frac{1}{3}\right)$$

$$= \log_2(3) \approx \underline{\underline{1,58}}$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b)$$
$$\log(a\cdot b) = \log(a) + \log(b)$$

$$\log(a^n) = n\cdot\log(a)$$

$$\log\left(\frac{1}{3}\right) = \log\left(3^{-1}\right) = (-1)\cdot\log(3)$$
$$\log\left(\frac{1}{3}\right) = \log(1) - \log(3) = 0 - \log(3)$$

Definition 6. *Let $X, Y \sim P_{X,Y}$. The mutual information of X and Y is given by*

$$I(X;Y) = D(P_{X,Y}\|P_X P_Y) = \sum_{x,y} P_{X,Y}(x,y)\log\frac{P_{X,Y}(x,y)}{P_X(x)P_Y(y)}$$

mutual information

e) $$I(X;Y) = \sum_{(x,y)\in\mathcal{X}\times\mathcal{Y}} p_{X,Y}(x,y)\cdot\log\frac{P_{X,Y}(x,y)}{P_X(x)\cdot P_Y(y)} =$$

X, Y unabhängig

$$\mathbb{P}(X=x, Y=y) \overset{\circ}{=} \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y)$$

Ex 1. 5 *Let $p(x,y)$ be given as in the table.*

Find

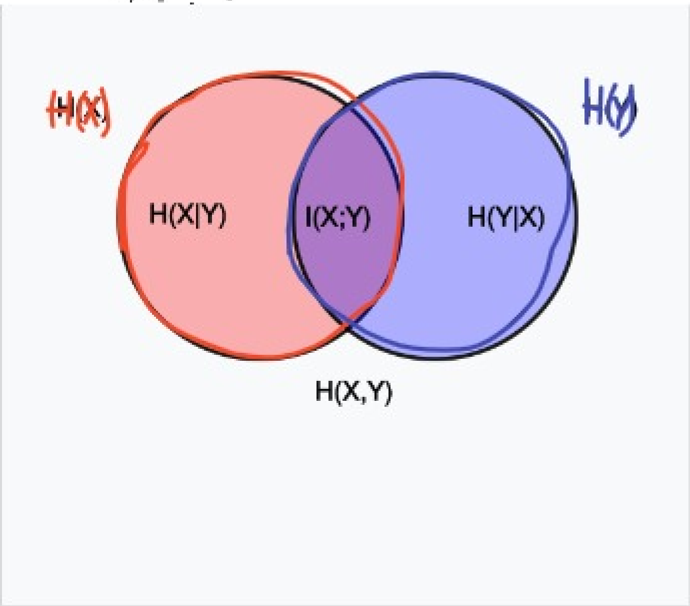
- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X,Y)$

$X \backslash Y$	0	1	
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
1	0	$\frac{1}{3}$	$\frac{1}{3}$
	$\frac{1}{3}$	$\frac{2}{3}$	

$$= \overset{\frac{1}{3}}{p_{X,Y}(0,0)}\cdot\log\frac{\frac{1}{3}}{\frac{2}{3}\cdot\frac{1}{3}} + \overset{\frac{1}{3}}{p_{X,Y}(1,1)}\cdot\log\frac{\frac{1}{3}}{\frac{1}{3}\cdot\frac{2}{3}} +$$

$+ \ldots$

- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X,Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X;Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*



Venn diagram showing additive and subtractive relationships among various information measures associated with correlated variables X and Y. The area contained by both circles is the joint entropy $H(X,Y)$. The circle on the left (red and violet) is the individual entropy $H(X)$, with the red being the conditional entropy $H(X|Y)$. The circle on the right (blue and violet) is $H(Y)$, with the blue being $H(Y|X)$. The violet is the mutual information $I(X;Y)$.

$$H(X|Y) + I(X,Y) = H(X)$$
$$H(Y|X) + I(X,Y) = H(Y)$$
$$H(X,Y) = H(X) + H(Y) - I(X,Y)$$

Ex 1. 4 *An urn contains r red, w white, and b black balls. Which has higher entropy, drawing $k \geq 2$ balls from the urn with replacement or without replacement? Set it up and show why. (There is both a hard way and a relatively simple way to do this.)*

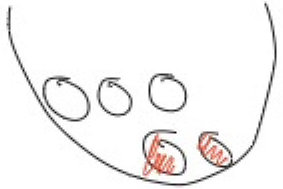
Example $k=2$:

X = outcome of first ball $\qquad X, Y \in \{r, w, b\}$

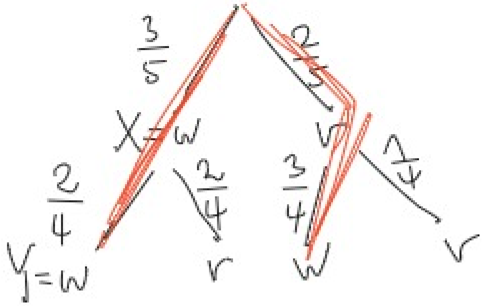
Y = outcome of second ball

With replacement $\Leftrightarrow$ X, Y independent $\Leftrightarrow$ $I(X,Y) = 0$

Symmetry with and without replacement: $H(X) = H(Y)$



$$\mathbb{P}(X=w) = \frac{3}{5}$$



$$\mathbb{P}(Y=w) = \frac{3}{5}\cdot\frac{2}{4} + \frac{2}{5}\cdot\frac{3}{4} = $$
$$= \cancel{2}\cdot\frac{2\cdot 3}{\cancel{4}\cdot 5} = \frac{3}{5} \Rightarrow H(X) = H(Y)$$

$$H(X,Y) = H(X) + \overset{\overset{H(X)}{\shortparallel}}{H(Y)} - I(X,Y)$$
$$= 2\cdot \underset{\substack{\uparrow \\ \text{same with} \\ \text{or without}}}{H(X)} - \underbrace{I(X,Y)}_{= \begin{cases} 0 & \text{without replacement} \\ >0 & \text{else} \end{cases}} = \Big\{$$

try : $H(X) = H(Y)$

ement

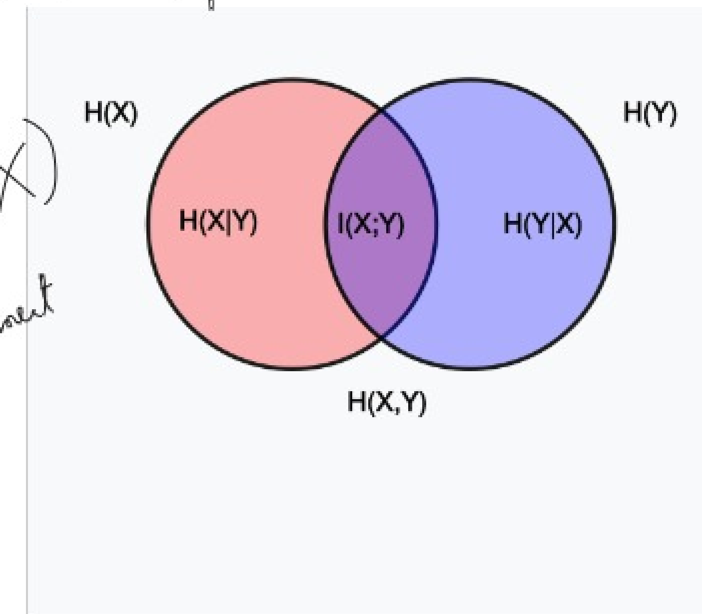
$$H(X,Y) = H(X) + \overset{\overset{H(X)}{\shortparallel}}{H(Y)} - \mathcal{I}(X,Y) \left\{ \begin{array}{ll} = 2H(X) & \text{without replacement} \\ < 2 \cdot H(X) & \text{with replacement} \end{array} \right.$$

$$= 2 \cdot H(X) - \underbrace{\mathcal{I}(X,Y)}_{= \begin{cases} 0 & \text{without replacement} \\ > 0 & \text{else} \end{cases}} =$$

same with or without replacement

Answer :

If we do the experiment with replacement , we have more incertainty or entropy.



Venn diagram showing additive and subtractive relationships among various information measures associated with correlated variables X and Y. The area contained by both circles is the joint entropy $H(X,Y)$. The circle on the left (red and violet) is the individual entropy $H(X)$, with the red being the conditional entropy $H(X|Y)$. The circle on the right (blue and violet) is $H(Y)$, with the blue being $H(Y|X)$. The violet is the mutual information $I(X;Y)$.

Ex 1.6 *Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.*

- *a= Show that $H(Z|X) = H(Y|X)$. Argue that if X, Y are independent, then $H(Y) \leq H(Z)$ and $H(X) \leq H(Z)$. Thus the addition of independent random variables adds uncertainty.*
- *b) Give an example of (necessarily dependent) random variables in which $H(Y) \geq H(Z)$and $H(X) \geq H(Z)$.*

$A = \{1,2,3\}$ $\quad f : A \times B \to \mathbb{R}$

$B = \{4,5\}$ $\quad (x,y) \mapsto x \cdot y$

$$\sum_{(x,y) \in A \times B} f(x,y) = 1 \cdot 4 + 1 \cdot 5 + 2 \cdot 4 + \ldots$$

$$A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$$

$$A \times B = \{(a,b) \mid a \in A, b \in B\}$$

$$\sum_{i=1}^{3} i = \sum_{i \in \{1,2,3\}} i$$

a) $$H(Z|X) = - \sum_{(z,x) \in \mathcal{Z} \times \mathcal{X}} p_{Z,X}(z,x) \cdot \log_2 \left(p_{Z|X}(z|x) \right)$$

$$= \sum_{(z,x) \in \mathcal{Z} \times \mathcal{X}} p_{Y,X}(z-x,x) \cdot \log_2 \left(p_{Y|X}(z-x|x) \right) = - \sum_{x \in \mathcal{X}} \sum_{z \in \mathcal{Z}} \ldots$$

$$H(Y|X) = - \sum_{(y,x) \in \mathcal{Y} \times \mathcal{X}} p_{Y,X}(y,x) \cdot \log_2 \left(p_{Y|X}(y|x) \right)$$

$$p_{Z,X}(z,x) = \mathbb{P}(Z=z, X=x)$$
$$= \mathbb{P}(X+Y=z, X=x)$$
$$= \mathbb{P}(Y=z-x, X=x)$$

$$p_{Z|X}(z|x) = \mathbb{P}(Z=z \mid X=x)$$
$$= \mathbb{P}(X+Y=z \mid X=x)$$
$$= \mathbb{P}(Y=z-x \mid X=x)$$
$$= p_{Y|X}(z-x|x)$$

$$\mathcal{Y} = \{ y \mid p_Y(y) > 0 \}$$

$$\mathcal{Z} = \{ z \mid p_Z(z) > 0 \} = \{ z \mid p_{X+Y}(z) > 0 \}$$
$$= \{ x+y \mid x \in \mathcal{X}, y \in \mathcal{Y} \}$$
$$= \bigcup_{x \in \mathcal{X}} \{ x+y \mid y \in \mathcal{Y} \} = \bigcup_{x \in \mathcal{X}} \bigcup_{y \in \mathcal{Y}} \{x+y\}$$

$$\mathcal{Z} = \bigcup_{z \in \mathcal{Z}} \{z\}$$

$$= - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{Y,X}(\overset{x+y}{\overset{\shortparallel}{z}} - x, x) \cdot \log \left(p_{Y|X}(\overset{x+y}{\overset{\shortparallel}{z}} - x | x) \right)$$

$$\Rightarrow H(Z|X) = H(Y|X)$$

miro